Casimir energy and the superconducting phase transition

Giuseppe Bimonte[1], Enrico Calloni, Giampiero Esposito and Luigi Rosa

INFN, Sezione di Napoli, and Dipartimento di Scienze Fisiche,
Complesso Universitario di Monte S Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

Abstract
We study the influence of Casimir energy on the critical field of a
superconducting film, and we show that by this means it might be possible
to directly measure, for the first time, the variation of Casimir energy
that accompanies the superconducting transition. It is shown that this
novel approach may also help clarify the long-standing controversy on the
contribution of TE zero modes to the Casimir energy in real materials.

PACS numbers: 03.70.+k, 74.20.−z, 74.25.−q, 74.76.Db

1. Introduction

The last ten years have witnessed intense experimental work on the Casimir effect [1]. The
terrific improvements in experimental techniques made it possible to measure the Casimir
force with an unprecedented precision, at the per cent level, with respect to the historical
experiments performed only a few decades ago. It seems fair enough to summarize the present
situation by saying that the experiments on the Casimir force have shown good quantitative
agreement with the theory, within the limits that are reasonable for experiments dealing with
macroscopic physics, and hence one may wonder what comes next. While there remain
important issues to be addressed (most notably that of thermal corrections in real materials)
which require further experimental refinements especially at large separations, we think the
time has come to search for entirely new directions of experimental activity on the physical
effects of vacuum fluctuations, going beyond force measurements. It occurred to us that no
experiments yet exist, which probe directly the physical effects of Casimir *energy*. Energy
is a more fundamental quantity than force, and therefore it seems to us that this would be a
rewarding target.

In view of the important role that the energy of vacuum fluctuations may have played
in the Early Universe, we considered two possible directions as interesting. One deals with
the gravitational effects of the Casimir energy, and indeed some time ago [2] we studied

[1] Talk given by this author.
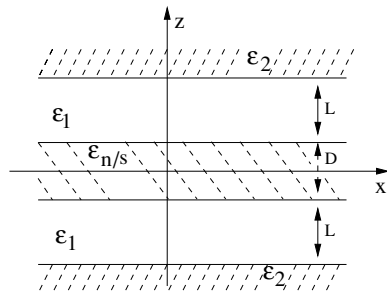
Figure 1. Scheme of the superconducting five-layer cavity.

the feasibility of an experiment aimed at verifying the validity of the equivalence principle of general relativity for the zero-point energy of vacuum fluctuations. While we are still working on this problem, the findings in [2] indicate that such an experiment might be feasible, provided that signal modulation problems can be solved. The second direction that we undertook concerns the influence of Casimir energy on phase transitions. We studied in particular the superconducting phase transition [3], and this contribution provides a summary of the work done so far. The results are very encouraging, and indeed the INFN has recently sponsored our experiment ALADIN2, to test the effects that are described in this paper. This represents a new approach to the Casimir effect that might contribute also to clarify some controversial issues regarding the Casimir effect in real materials.

The plan of the paper is as follows: in section 2 we present the general theoretical ideas involved in our experiment[2], while section 3 explains how to use Lifshitz theory to compute the variation of Casimir energy across the superconducting transition. In section 4, we present the results of our numerical computations, and in section 5 we examine the issue of the contribution of the TE zero mode. Finally, section 6 contains our conclusions and a discussion of the results.

2. The Casimir effect in a superconducting cavity

Consider the double cavity shown in figure 1, obtained by placing a thin superconducting film, with thickness D, between the plates of a rigid plane-parallel Casimir cavity. The two gaps at the sides of the film, of common width L, are filled with some insulator. It is well known that the magnitude of the Casimir effect depends on the reflective power of the layers forming the cavity. Now, experiments show [4] that the reflective properties of a superconducting film, in the microwave region of the spectrum, are drastically different from those in the normal state. Therefore one can foresee that both the Casimir free energy stored in the cavity and the Casimir force on the outer plates change when the film passes from normal (n) to superconducting (s), and one wonders if there is a way to measure these effects. A standard force measurement on the outer plates would be certainly impractical, because the variation of the Casimir force across the transition is extremely small [5]. The reason for this is easy to understand, and is due to the fact that the transition to superconductivity affects the reflective power of the film only at wavelengths of order $c\hbar/(kT_{\rm c}) \simeq 2$ mm (for a typical critical temperature $T_{\rm c} \simeq$ 1 K), which are very far from the submicron range, that gives the dominant contribution to the Casimir force for typical Casimir cavities. In fact, we estimated that in typical conditions the

---

[2] The experimental aspects of the setup are discussed in a separate paper in this issue.

relative variation of the Casimir force is of order 10^{-8} or less, which is clearly immeasurable within the present level of precision, which is only of a few per cent. Therefore, one has to consider alternative effects, and we realized that a feasible scheme involves the measurement of the critical magnetic field required to destroy the superconductivity of the film. Let us see why this new approach may very well work.

As is well known, superconductors show perfect diamagnetism, as they expel magnetic fields from their interior. However, this is true only for magnetic fields not exceeding a critical value H_c, above which it becomes energetically convenient for the superconductor to revert to the normal state and let the magnetic field in. For standard samples, the value of H_c can be obtained by equating the magnetic work W, done to expel the critical field, to the so-called condensation energy of the superconductor $\mathcal{E}_{cond}(T)$, defined as the difference among the Helmholtz free energies of the film, in the n and in the s state [6]. For a thick film (with thickness D much greater than the superconductor penetration depth λ and correlation length ξ) and a parallel field, one finds $W = V H^2/(8\pi)$, with V the volume of the film, and therefore one gets for $H_{c\parallel}$ the equation:

$$V \frac{H_{c\parallel}^2}{8\pi} = \mathcal{E}_{cond}(T). \tag{1}$$

What happens if the film is placed now inside a Casimir cavity? With respect to the previous situation, we have to take into account that the magnetic work W must now be balanced against the condensation energy of the film *plus* the difference $\Delta F_E^{(C)}(T) = F_n^{(C)}(T) - F_s^{(C)}(T)$ between the Casimir free energy $F_{n/s}^{(C)}(T)$ in the n/s states of the film, respectively, and therefore one obtains the following modified equation for the critical field:

$$V \frac{\left(H_{c\parallel}^{cav}\right)^2}{8\pi} = \mathcal{E}_{cond}(T) + \Delta F_E^{(C)}(T). \tag{2}$$

On intuitive grounds, we expect $\Delta F_E^{(C)}(T)$ to be a positive quantity, because in the superconducting state the film should be closer to behaving as an ideal mirror, than in the normal state, and therefore $F_s^{(C)}(T)$ should be more *negative* than $F_n^{(C)}(T)$. In view of equation (2), this implies that the critical field should be shifted by the Casimir term towards larger values. Upon comparing equations (1) and (2), we see the shift of critical field should have a relative magnitude approximately equal to

$$\delta H_{c\parallel}/H_{c\parallel} \approx \Delta F_E^{(C)}(T) \big/ (2\mathcal{E}_{cond}(T)). \tag{3}$$

The key point to note here is that the relative shift of critical field is determined by the ratio of $\Delta F_E^{(C)}(T)$ not to $F_{n/s}^{(C)}(T)$ (as explained earlier this ratio is going to be very small indeed) but rather to the condensation energy of the film. Now, the latter quantity is very small for a thin film, and for a film thickness of a few nm,[3] it is easily several orders of magnitude smaller than typical Casimir energies. Therefore, even if $\Delta F_E^{(C)}(T)$ is a tiny fraction of the Casimir energy of the cavity, it may be easily comparable to $\mathcal{E}_{cond}$, and produce a measurable shift of critical field. In fact, in the case of a Be film, we estimated that a relative variation of Casimir energy as small as one part in 10^8, could still correspond, close to T_c, to more than 10% of $\mathcal{E}_{cond}$ and would therefore induce a shift of critical field of over 5%!

---

[3] For such ultrathin films, equations (1) and (2) above should be modified to take account of incomplete field expulsion in thin films. However, the final formula for the relative shift equation (3) remains unaltered. See [3] for the details.

3. Computation of $\Delta F_E^{(C)}(T)$

We have evaluated $\Delta F_E^{(C)}(T)$ by means of Lifshitz theory for the Casimir effect in dispersive media [7]. We recall that the fundamental physical assumption of that theory is that one can describe, in the relevant range of frequencies, the propagation of electromagnetic waves in the material bodies forming the cavity, by means of a complex permittivity $\epsilon(\omega)$, depending only on the frequency ω and not on the wave-vector **q**. Therefore, Lifshitz theory cannot be applied in situations where space dispersion becomes important. In our case, the applicability of such a theory to the computation of $\Delta F_E^{(C)}(T)$ might be questioned, because the characteristic wavelengths which occur in the determination of $\Delta F_E^{(C)}(T)$, as pointed out earlier, belong to the microwave region of the spectrum, where normal metals may show an anomalous skin effect. This is especially true at cryogenic temperatures, when the anomalous region further extends, due to longer electron's mean free paths. In the superconducting state of the film, non-local effects may be even more important because, due to the small skin depth of electromagnetic fields in superconductors, the anomalous skin effect is observed, in clean superconductors, even inside the frequency domain characteristic of the normal skin effect in normal metals (extreme anomalous skin effect). Fortunately, however, non-local effects are less important in ultrathin films (with thickness D much smaller than the penetration depth λ), than in bulk samples. The reason is that the electron mean free paths in ultrathin films cannot be very large, even in the superconducting state. For example, the authors of [8] quote a mean free path of 64 nm in pure ultrathin superconducting Be films with a thickness $D = 4.2$ nm ($T_c = 0.6$ K). Therefore, when considering ultrathin films, one is in the so-called dirty case, where local electrodynamics remains a valid approximation. This is confirmed by experiments [4], showing that the film conductivity is independent of film thickness, for small thicknesses.

Let us briefly recall how to compute $\Delta F_E^{(C)}(T)$. As is well known, there exists a simple derivation of the Lifshitz formula for the Casimir energy in dispersive media, based on consideration of the stationary modes of the cavity [9]. This approach is best suited to study our five-layer system (see figure 1), for which the original Lifshitz method would be very complicated. The electric permittivities of the layers are denoted as follows: $\epsilon_{n/s}$ represents the permittivity of the film, in the n/s states respectively, while ϵ_1 is the permittivity of the insulating layers. Last, ϵ_2 is the permittivity of the outermost thick normal metallic plates. According to the mode method, one can write the unrenormalized variation of Casimir energy $\Delta E_0^{(C)}$, at $T = 0$, as

$$\Delta E_0^{(C)} = \frac{\hbar A}{2} \int \frac{\mathrm{d}k_1 \, \mathrm{d}k_2}{(2\pi)^2} \sum_{\alpha=\text{TE,TM}} \sum_p \left(\omega_{\mathbf{k}_\perp,p}^{(n,\alpha)} - \omega_{\mathbf{k}_\perp,p}^{(s,\alpha)} \right), \tag{4}$$

where $A \gg L^2$ is the area, $\mathbf{k}_\perp = (k_1, k_2)$ denotes the two-dimensional wave vector in the xy plane, while $\omega_{\mathbf{k}_\perp,p}^{(n/s,\text{TM})}$ $\left(\omega_{\mathbf{k}_\perp,p}^{(n/s,\text{TE})} \right)$ denote the proper frequencies of the TM (TE) modes, in the n/s states of the film, respectively. Upon using the Cauchy theorem (for details, we address the reader to chapter 4 in [10])[4], we can rewrite the sums in equation (4) as integrals over *complex* frequencies iζ:

$$\left(\sum_p \omega_{\mathbf{k}_\perp,p}^{(n,\text{TM})} - \sum_p \omega_{\mathbf{k}_\perp,p}^{(s,\text{TM})} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\zeta \, \log \frac{\Delta_n^{(1)}(\mathrm{i}\zeta)}{\Delta_s^{(1)}(\mathrm{i}\zeta)}, \tag{5}$$

---

[4] When comparing the formulae of this paper with those of [10], please note that our L and D correspond, respectively, to d and a of [10], while the TM and TE modes are labelled there by the suffices (1) and (2), respectively. Note also that in our configuration the central layer is constituted by the superconducting film, and not by the vacuum, and hence its permittivity, denoted by ϵ_0 in [10], is not equal to 1, but rather to $\epsilon_{n/s}$ depending on the state of the film.

where $\Delta_{n/s}^{(1)}(\mathrm{i}\zeta)$ is the expression in equation (4.7) of [10] (evaluated for $\epsilon_0 = \epsilon_{n/s}$). A similar expression can be written for the TE modes, which involves the quantity $\Delta_{n/s}^{(2)}(\mathrm{i}\zeta)$ defined in equation (4.9) of [10]. It is interesting to note that the integral on the rhs of equation (5) is *finite* because, as observed earlier, the ratio $\Delta_n^{(1)}(\mathrm{i}\zeta)\big/\Delta_s^{(1)}(\mathrm{i}\zeta)$ is appreciably different from one only for frequencies ζ in the microwave region (the same is true for the TE contribution). Therefore, there is no need here for an infinite renormalization, contrary to what usually happens when evaluating Casimir energies. There is however a *finite* subtraction to perform, because we require that the variation of Casimir energy $\Delta E^{(C)}$ should approach zero for infinite separations L. Upon subtracting from equation (5) (and the analogous expression for TE modes) its limiting value for $L \to \infty$, and after performing the change of variables $k_\perp^2 = (p^2 - 1)\zeta^2/c^2$ in the integral over $k_\perp$, one gets the following expression for the (renormalized) variation of Casimir energy,

$$\Delta E^{(C)} = \frac{\hbar A}{4\pi^2 c^2} \int_1^\infty p\, \mathrm{d}p \int_0^\infty \mathrm{d}\zeta\, \zeta^2 \sum_{\alpha=\mathrm{TE,TM}} \log \frac{Q_n^\alpha}{Q_s^\alpha}, \qquad (6)$$

where

$$Q_I^\alpha(\zeta, p) = \frac{\left(1 - \Delta_{1I}^\alpha \Delta_{12}^\alpha\, \mathrm{e}^{-2\zeta K_1 L/c}\right)^2 - \left(\Delta_{1I}^\alpha - \Delta_{12}^\alpha\, \mathrm{e}^{-2\zeta K_1 L/c}\right)^2 \mathrm{e}^{-2\zeta K_I D/c}}{1 - \left(\Delta_{1I}^\alpha\right)^2 \mathrm{e}^{-2\zeta K_I D/c}} \qquad (7)$$

and

$$\Delta_{jl}^{\mathrm{TE}} = \frac{K_j - K_l}{K_j + K_l}, \qquad \Delta_{jl}^{\mathrm{TM}} = \frac{K_j \epsilon_l(\mathrm{i}\zeta) - K_l \epsilon_j(\mathrm{i}\zeta)}{K_j \epsilon_l(\mathrm{i}\zeta) + K_l \epsilon_j(\mathrm{i}\zeta)}, \qquad (8)$$

$$K_j = \sqrt{\epsilon_j(\mathrm{i}\zeta) - 1 + p^2} \qquad I = n, s, \qquad j, l = 1, 2, n, s. \qquad (9)$$

The extension of the above formulae to the case of finite temperature is straightforward. As is well known this amounts to the replacement in equation (6) of the integration $\int \mathrm{d}\zeta/2\pi$ by the summation $kT/\hbar \sum_l$ over the Matsubara frequencies $\zeta_l = 2\pi l/\beta$, where $\beta = \hbar/(kT)$, which leads to the following expression for the variation $\Delta F_E^{(C)}(T)$ of Casimir free energy:

$$\Delta F_E^{(C)}(T) = A\frac{kT}{2} \sum_{l=-\infty}^\infty \int \frac{\mathrm{d}\mathbf{k}_\perp}{(2\pi)^2} \left(\log \frac{Q_n^{\mathrm{TE}}}{Q_s^{\mathrm{TE}}} + \log \frac{Q_n^{\mathrm{TM}}}{Q_s^{\mathrm{TM}}}\right). \qquad (10)$$

As we see, equations (6)–(10) involve the electric permittivities $\epsilon(\mathrm{i}\zeta)$ of the various layers at imaginary frequencies $\mathrm{i}\zeta$. For these functions, we have made the following choices.

For the outermost metal plates, we use a Drude model for the electric permittivity:

$$\epsilon_{\mathrm{D}}(\omega) = 1 - \frac{\Omega^2}{\omega(\omega + \mathrm{i}\gamma)}, \qquad (11)$$

where Ω is the plasma frequency and $\gamma = 1/\tau$, with τ the relaxation time. We denote by Ω_2 and τ_2 the values of these quantities for the outer plates. As is well known, the Drude model provides a very good approximation in the low-frequency range $\omega \approx 2kT_{\mathrm{c}}/\hbar \simeq 10^{11}$–$10^{12}$ rad s^{-1} which is involved in the computation of $\Delta F_E^{(C)}(T)$ and $\Delta E^{(C)}$. The relaxation time is temperature dependent and for an ideal metal it becomes infinite at $T = 0$. However, in real metals, the relaxation time stops increasing at sufficiently low temperatures (typically of order a few K), where it reaches a saturation value, which is determined by the impurities that are present in the metal. Since in a superconducting cavity the temperatures are very low, we can assume that τ_2 has reached its saturation value and therefore we can treat it as a constant. The continuation of equation (11) to the imaginary axis is of course straightforward and gives

$$\epsilon_{\mathrm{D}}(\mathrm{i}\zeta) = 1 + \frac{\Omega^2}{\zeta(\zeta + \gamma)}. \qquad (12)$$
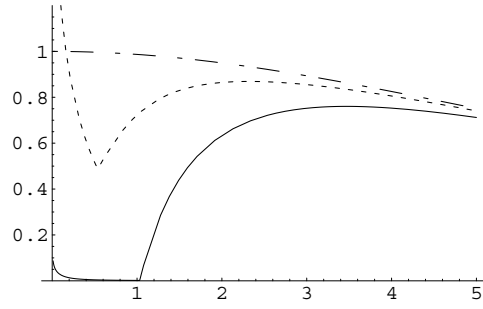
Figure 2. Plots of $m\sigma_s'(\omega)/(ne^2\tau_n)$, for $T/T_c = 0.3$ (solid line), $T/T_c = 0.9$ (dashed line) and $T = T_c$ (point-dashed line). On the abscissa, the frequency ω is in reduced units $x_0 = \hbar\omega/(2\Delta(0))$ and $y_0 = 2\Delta(0)/\tau_n \simeq 8.7$.

For the insulating layers, we take a constant permittivity, equal to the static value:

$$\epsilon_1(\omega) = \epsilon_1(0). \tag{13}$$

Again, this is a good approximation in the range of frequencies that we consider.

As for the film, in the normal state we use again the Drude expression equation (11), with appropriate values for the plasma frequency Ω_n and the relaxation time τ_n.

The permittivity $\epsilon_s(i\zeta)$ of the film in the superconducting state cannot be given in closed form and we have evaluated it by using the Mattis–Bardeen formula for the conductivity $\sigma_s(\omega)$ of a superconductor in the local limit $q \to 0$ of BCS theory. Actually, for the evaluation of $\Delta F_E^{(C)}(T)$ we need only consider the real part $\sigma_s'(\omega)$ of the complex conductivity, because the expression of the permittivity $\epsilon_s(i\zeta)$ along the imaginary axis, which occurs in the Lifshitz formula, can be obtained from that of $\sigma_s'(\omega)$ by use of the dispersion relation

$$\epsilon_s(i\zeta) - 1 = 8 \int_0^\infty d\omega \, \frac{\sigma_s'(\omega)}{\zeta^2 + \omega^2}. \tag{14}$$

The reader can find explicit formulae for $\sigma_s'(\omega)$ in [3]. Here, we observe only that $\sigma_s'(\omega)$ can be thought of as the sum of three contributions: a δ function at the origin, a broad thermal component that diverges logarithmically at $\omega = 0$ and a direct absorption component, with an onset at $2\Delta(T)$. At any $T < T_c$, complete specification of $\sigma_s'(\omega)$ requires fixing three parameters: besides the free electron density n (or equivalently the square of the plasma frequency $\Omega_n^2 = 4\pi ne^2/m$) that provides the overall scale of σ_s', and the relaxation time for the normal electrons τ_n, both of which already occur in the simple Drude formula, $\sigma_s'(\omega)$ only depends on one extra parameter, i.e. the gap Δ. We point out that this expression for σ_s' is valid for arbitrary relaxation times τ_n, i.e. for arbitrary mean free paths, and in particular it holds in the so-called impure limit $y = 2\Delta/(\hbar\tau_n) \gg 1$, where the effects of non-locality become negligible.

We point out that at fixed ω for $T \to T_c$, as well as at fixed $T < T_c$ for $x \equiv \hbar\omega/(2\Delta) \to \infty$, $\sigma_s'(\omega)$ approaches the Drude expression $\sigma_D'(\omega)$

$$\sigma_D'(\omega) = \frac{1}{4\pi} \frac{\Omega^2\tau}{1 + \omega^2\tau^2}. \tag{15}$$

The convergence of $\sigma_s'(\omega)$ to $\sigma_D'(\omega)$ in the frequency domain is in fact very fast, and already for x of order 10 or so σ_s' becomes undistinguishable from σ_D', in accordance with experimental findings [4]. In figure 2 we show the plots of $\sigma_s'(\omega)m/(ne^2\tau_n)$, for three values of the reduced temperature $t \equiv T/T_c$, i.e. $t = 0.3, 0.9$ and $t = 1$. The curves are computed for $y_0 = 2\Delta(0)/\tau_n \simeq 8.7$. Frequencies are measured in reduced units $x_0 = \hbar\omega/(2\Delta(0))$.
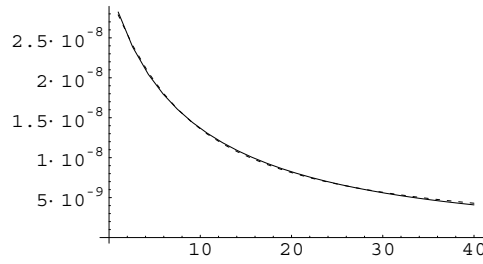
Figure 3. Plots of $\Delta F_E^{(C)}$ (in erg) as a function of L (in nm) for $D = 5$ nm, $T_c = 0.5$ K, $t = 0.9$. See text for the values of the other parameters. Also shown is the plot (dashed line) of a fit of the type $1/(1 + (L/L_0)^\alpha)$, with $L_0 = 8.3$ nm and $\alpha = 1.15$.

4. Results

We have evaluated numerically $\Delta F_E^{(C)}(T)$, and the results of the computation can be summarized as follows:

- The contribution of TM modes to $\Delta F_E^{(C)}(T)$ is negligible with respect to that of TE modes (by three orders of magnitude or so).
- $\Delta F_E^{(C)}(T)$ is practically independent (to better than four digits) of the value of the dielectric constant of the insulating gaps.
- $\Delta F_E^{(C)}(T)$ increases with film thickness D and saturates for $D \simeq c/\Omega_p \simeq 10$ nm.
- $\Delta F_E^{(C)}(T)$ increases when the gap separation L decreases, and approaches a finite limit, for $L \to 0$.
- $\Delta F_E^{(C)}(T)$ increases significantly with the plasma frequency of the film Ω_n and of the outer plates Ω_2.
- $\Delta F_E^{(C)}(T)$ has a maximum for values around 10 of the impurity parameter y.

In figure 3 we show the plot of $\Delta F_E^{(C)}$ (in erg) as a function of the width L (in nm) of the insulating gap, for $D = 5$ nm, $T_c = 0.5$ K, $y = 15$, $t = 0.9$, $\Omega_n = \Omega_2 = 18.9$ eV, $\tau_2 = 2.4 \times 10^{-12}$ s. We observe that $\Delta F_E^{(C)}$ is always *positive*, which corresponds to the intuitive expectation that transition to superconductivity of the film leads to a *stronger* Casimir effect, i.e. to *lower* Casimir free energy. The data can be fit very accurately by a curve of the type

$$\Delta F_E^{(C)}(L) \propto \frac{1}{1 + (L/L_0)^\alpha}, \tag{16}$$

where $L_0 = 8.3$ nm and $\alpha = 1.15$.

In figure 4 we show (solid line) the relative shift of the critical parallel field of a Be film, as a function of t. The figure has been drawn by using the same parameters as in figure 3. Note that the shift is *positive*, meaning that the critical field for the film placed in the cavity is *larger* than the critical field for an identical film outside the cavity. The increase of critical field relative shift as one approaches the critical temperature arises because, for $t \to 1$, $\Delta F_E^{(C)}$ and $\mathcal{E}_{cond}$ approach zero at different rates. Indeed, while $\mathcal{E}_{cond}$ vanishes as $(1 - t)^2$, $\Delta F_E^{(C)}$ is found to vanish approximately like the first power of $1 - t$.

5. Contribution from the TE zero mode

In recent years many efforts have been made to compute the combined influence of temperature and finite conductivity of the plates on the Casimir effect, and no agreement has been reached
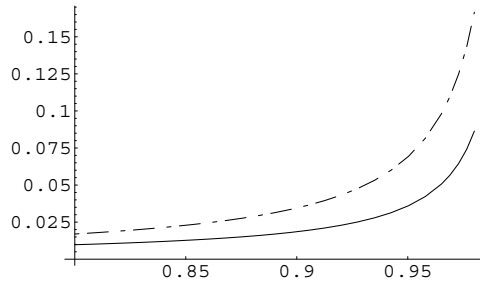
Figure 4. Plot (solid line) of the relative shift of parallel critical field for a Be cavity, as a function of T/T_c. The point-dashed line has been computed using the plasma model for the outer plates. Cavity parameters are as in figure 3.

among the experts on the proper way of calculating the contribution of the TE zero mode (i.e. the $l = 0$ term in the Matsubara sum) to the Casimir free energy (see [11] for a discussion of different points of view on this problem). This is a delicate issue because, according to Lifshitz theory, the computation of this mode involves the quantity

$$C := (\zeta^2 \epsilon(\mathrm{i}\zeta))|_{\zeta=0}. \tag{17}$$

The problem is that, in the Matsubara formalism where ζ is discrete, C is ill defined if $\epsilon(\mathrm{i}\zeta)$ diverges at $\zeta = 0$, which is the case for metals, and then the results depend on how one resolves the ambiguity. If ζ is viewed as a continuous variable, one may *define* C as the limit of $(\zeta^2 \epsilon(\mathrm{i}\zeta))$ for vanishing ζ. In this case, if one uses for the metal conductivity the Drude model equation (12) (with a finite value of τ), one obtains $C = 0$, and this implies that the TE zero mode gives zero contribution to the Casimir free energy, irrespective of how large the relaxation time τ is. The odd thing is that the result is different if, instead of the Drude model, one uses the simpler plasma model

$$\epsilon(\mathrm{i}\zeta) = 1 + \frac{\Omega_p^2}{\zeta^2}, \tag{18}$$

for then one finds $C = \Omega_p^2$, and therefore the zero mode gives a non-vanishing contribution, reproducing the ideal metal case in the limit of infinite plasma frequency.

It is clear that in such a situation it would be very interesting to have the possibility of an experimental verification of these effects, and we show below that a superconducting cavity is very well suited for this purpose.

The computations presented in the previous section were performed by using the Drude model, both for the lateral plates and the film (in the normal state), and therefore the computed value of $\Delta F_E^{(C)}$ receives no contribution from the TE zero mode. We have repeated the computation, by using this time the plasma model for the lateral plates and we denote by $\widetilde{\Delta F}_E^{(C)}$ the corresponding value of the variation of Casimir energy. Note that in this new computation, we keep the Drude model for the film in the normal state, because this is the limit of the BCS expression of the permittivity for $T \rightarrow T_c$. In figure 5 we plot (dashed line) $\widetilde{\Delta F}_E^{(C)}$ as a function of t. We point out that the inclusion of the TE zero mode has the largest effect close to T_c, where it leads to an approximate doubling in the value of $\Delta F_E^{(C)}$. The reason of this is easy to understand: the zero mode becomes more and more important in the critical region, because a decreasing number of Matsubara modes contribute to $\Delta F_E^{(C)}$, as one approaches T_c, and therefore the inclusion or omission of a single mode makes a big difference. That less and less modes contribute as we move towards T_c is clear, because
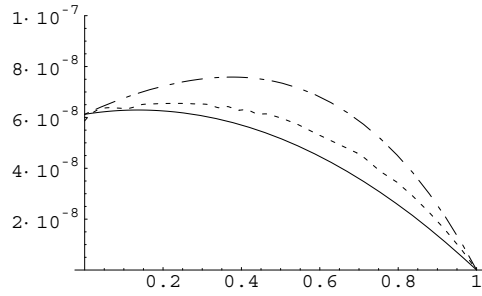
Figure 5. Plot (solid line) of $\Delta F_E^{(C)}$ (in erg) as a function of t for $L = 10$ nm, $D = 5$ nm, $T_{\rm c} = 0.5$ K, $\tau_n = 5 \times 10^{-13}$ s. The point-dashed line was computed by using the plasma model for lateral plates. Also shown (dashed line) is the plot of the low-temperature limit of the Matsubara sum, equation (6). See text for further details.

the quantities $\left(Q_n^{\rm TE}/Q_s^{\rm TE}\right)(i\zeta)$ in equation (10) are substantially different from one only for complex frequencies ζ of order a few times $kT_{\rm c}/\hbar$. Since the lth Matsubara mode has a frequency equal to $2\pi lkT/\hbar$, it is clear that the number of terms effectively contributing to $\Delta F_E^{(C)}$ should be roughly proportional to $T_{\rm c}/T$, and hence it is large for $T \ll T_{\rm c}$, but becomes small for T comparable to $T_{\rm c}$. In figure 4 we show (point-dashed line) the shift of critical field resulting from the plasma model, and we see that the amount of shift is almost doubled with respect to that resulting from use of the Drude model. It seems, therefore, that if the shift can be detected, it should be rather easy to distinguish among the two possibilities.

6. Conclusions and discussion

We have proposed a novel experimental approach to explore the physical effects of vacuum fluctuations of the electromagnetic field, based on the use of superconducting Casimir cavities. In our scheme, the object of interest is the Casimir energy itself, rather than the Casimir force, as in all experiments performed so far. We have shown that the superconducting transition of a thin film placed between the plates of a plane-parallel cavity, determines a small variation of Casimir energy. While the associated variation of the Casimir force on the outer plates is immeasurably small, we have found that there is a measurable effect on the critical magnetic field required to destroy the film superconductivity. Because of the Casimir effect, the critical field is larger than that of a similar film, not placed inside the cavity. The amount of the shift depends on the temperature, on the geometric features of the cavity, and on the materials chosen for the film and for the outer plates, and can be of the order of a few per cent.

The results presented in this paper represent the initial steps of a more general experimental research programme, on the influence of vacuum fluctuations on phase transitions. This is a new direction in the field of the Casimir effect that may contribute to a better understanding of the general issue of the rôle of vacuum energy in phase transitions, which is of great interest in diverse areas of physics, but most notably in cosmology.

We would also like to point out another couple of interesting features of our approach. One is that we use *rigid* cavities, which may represent an advantage over conventional Casimir experiments. As is well known, the experimental difficulty of controlling the parallelism among macroscopic plane plates with submicron separations led the experimenters to consider simpler geometries that do not suffer from this problem, such as the sphere-plane one, which has been adopted in all precision experiments on the Casimir force (with the only exception

of the experiment by Bressi *et al* [1], where the plane-parallel configuration is used, at the price, however, of a reduced precision compared with the sphere-plate case). This limitation has made it impossible so far to explore experimentally one of the distinguishing features of the Casimir effect, i.e. its dependence on the geometry of the cavity. The use of rigid cavities might make it possible to study geometries that are difficult to realize by using non rigid cavities.

Another interesting feature of our scheme relates to the current interest in the study of the Casimir effect in real materials, in particular for what concerns the influence of temperature and of the finite conductivity of the materials. In standard Casimir force measurements, it is generally quite difficult to measure these effects, because they typically represent small corrections to the ideal case, and therefore they are difficult to extract from the signal. In our setting, however, the effect is null if the film is treated as an ideal metal, and therefore the signal arises entirely from the fact that the film is treated as a real material, with different finite conductivities in the normal and in the superconducting state. Therefore, our approach seems best suited to test our understanding of the Casimir effect in real materials. As an example, we discussed the contribution from the TE zero mode. This is a controversial issue in the current literature on thermal corrections to the Casimir effect. It is well known that, for submicron plate separations, thermal corrections to the Casimir force are negligible at cryogenic temperature, and become relevant only at room temperature. However, things are different in our case, because close to $T_{\rm c}$ the shift of critical field is completely determined by the few Matsubara modes with frequencies below or of order $kT_{\rm c}/\hbar$, which is where the reflective properties of a film change when it becomes superconducting. As a consequence, different treatments of the TE zero mode lead to strongly different predictions for the shift of critical field, at the level of doubling the shift, and this opens the way to a possible experimental clarification of this delicate problem.

The verification of the effects described in this paper is the goal of the ALADIN2 experiment, financed by INFN, which is currently under way at the Dipartimento di Scienze Fisiche dell'Università di Napoli Federico II. Further details on this experiment can be found in a separate contribution, contained in this issue.

Acknowledgments

References

[1] Lamoreaux S K 1997 *Phys. Rev. Lett.* **78** 5
    Mohideen U and Roy A 1998 *Phys. Rev. Lett.* **81** 4549
    Bressi G, Carugno G, Onofrio R and Ruoso G 2002 *Phys. Rev. Lett.* **88** 041804
    Decca R S, López D, Fischbach E and Krause D E 2003 *Phys. Rev. Lett.* **91** 050402
    Chan H B, Aksyuk V A, Kleiman R N, Bishop D J and Capasso F 2001 *Science* **291** 1941
[2] Bimonte G, Calloni E, Di Fiore L, Esposito G, Milano L and Rosa L 2006 On the behaviour of a rigid Casimir cavity in a gravitational field *Recent Developments in Gravitational Physics (IOP Conference Series* vol 176*)* ed I Ciufolini *et al* (London: Taylor and Francis) (*Preprint* hep-th/0302082)
    Calloni E, Di Fiore L, Esposito G, Milano L and Rosa L 2002 *Phys. Lett.* A **297** 328
[3] Bimonte G, Calloni E, Esposito G, Milano L and Rosa L 2005 *Phys. Rev. Lett.* **94** 180402
    Bimonte G, Calloni E, Esposito G and Rosa L 2005 *Nucl. Phys.* B **726** 441

[4] Glover R E III and Tinkham M 1957 *Phys. Rev.* **108** 243
[5] Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and Its Applications* (Oxford: Clarendon)
[6] Tinkham M 1996 *Introduction to Superconductivity* (Singapore: McGraw-Hill)
[7] Lifshitz E M 1956 *Sov. Phys.—JETP* **2** 73
      Lifshitz E M and Pitaevskii L P 1980 *Landau and Lifshitz Course of Theoretical Physics: Statistical Physics:
           Part II* (Burlington, MA: Butterworth-Heinemann)
[8] Adams P W, Herron P and Meletis E I 1998 *Phys. Rev.* B **58** R2952
[9] Schram K 1973 *Phys. Lett.* A **43** 282
[10] Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
[11] Geyer B, Klimchitskaya G L and Mostepanenko V M 2003 *Phys. Rev.* A **67** 062102
      Høye J S, Brevik I, Aarseth J B and Milton K A 2003 *Phys. Rev.* E **67** 056116
      Decca R S, Lopez D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M 2005 *Ann. Phys.,
           NY* **318** 307 and references therein